Advanced General Relativity : Exercises sheet (2021-2022)

Exercise 1

In a space-time with torsion, one has that

$$\Gamma_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho}(g) - K_{\mu\nu}{}^{\rho} \ , \tag{1}$$

where $\Gamma_{\mu\nu}{}^{\rho}(g)$ are the torsion-free Christoffel symbols and $K_{\mu\nu}{}^{\rho}$ is the so-called *contorsion* tensor. Show that

$$\int d^4x \, \sqrt{-|g|} \, \nabla_{\mu} V^{\mu} \ , \tag{2}$$

is *not* a total derivative. Instead, it can be expressed as a boundary term plus an extra piece depending on the contorsion tensor, thus spoiling the usual integration by parts.

Note: $\partial_{\mu}\sqrt{-|g|} = \sqrt{-|g|}\,\Gamma_{\rho\mu}{}^{\rho}(g)$

Exercise 2

Let us consider the action of a spin 0 (scalar) field

$$S_{\phi} = \int d^4x \, \sqrt{-|g|} \, \left(-\frac{1}{2} \, F_{\mu} \, F^{\mu} - V(\phi) \right) \ , \tag{3}$$

with field strength

$$F_{\mu} = \nabla_{\mu}\phi = \partial_{\mu}\phi \ , \tag{4}$$

and a general scalar potential $V(\phi)$.

Assuming a space-time without torsion:

a) Compute the equation of motion for the scalar field ϕ.

b) Compute the stress-energy tensor $T_{\mu\nu}$.

c) Compute $T \equiv T_{\mu}{}^{\mu}$.

Exercise 3

Let us consider the action of a spin 1 (Maxwell) field

$$S_{A} = \int d^4x \, \sqrt{-|g|} \, \left(-\frac{1}{2 \cdot 2!} \, F_{\mu\nu} \, F^{\mu\nu} \right) \ , \tag{5}$$

with field strength

$$F_{\mu\nu} = \nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu} \ . \tag{6}$$

Assuming a space-time without torsion:

a) Compute the equation of motion for the Maxwell field A_{μ}.

b) Compute the stress-energy tensor $T_{\mu\nu}$.

c) Compute $T \equiv T_\mu{}^\mu$.

d) Using spherical coordinates $x^\mu = \{t, r, \theta, \varphi\}$, compute the explicit form of the stress-energy tensor $T_\mu{}^\nu$ for a Maxwell field of the form

$$A_\mu = \left(\frac{r_Q}{r}\,, \vec{0}\right) \;, \tag{7}$$

where $r_Q = \text{cst}$.

Exercise 4

Let us consider a simple wormhole metric of the form

$$ds^2 = -dt^2 + du^2 + (b_0^2 + u^2)(d\theta^2 + \sin^2\theta\, d\phi^2) \;, \tag{8}$$

in terms of a constant parameter $b_0 > 0$. The ranges of the coordinates are given by

$$t \in (-\infty, \infty) \quad , \quad u \in (-\infty, \infty) \quad , \quad \theta \in [0, \pi] \quad , \quad \phi \in [0, 2\pi] \;. \tag{9}$$

a) Using cylindrical coordinates in the embedding space

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2 \;, \tag{10}$$

construct an embedding diagram

$$z(r) \qquad \text{with} \qquad r^2 = b_0^2 + u^2 \;, \tag{11}$$

for the equatorial plane $\theta = \pi/2$ at a fixed time t . How many asymptotic regions does the geometry have?

b) Using the Einstein equations, compute the stress tensor $T_{\mu\nu}$ compatible with the metric (8). What kind of matter would this stress tensor be accounting for? Is it an ordinary type of matter?

Exercise 5

Let us consider the stress-energy tensor of a perfect fluid to be of the form

$$T_{\mu\nu} = -\frac{\Lambda}{\kappa^2} g_{\mu\nu} \qquad \text{with} \qquad \Lambda > 0 \;. \tag{12}$$

a) Discuss the equation of state $f(\rho, P) = 0$ compatible with a stress-energy tensor of the type in (12). Do you identify some pathology with this equation of state?

Note: The constant Λ is known as the *cosmological constant.*

b) Consider a scalar field ϕ in presence of a scalar potential $V(\phi)$ as a possible origin of the above $T_{\mu\nu}$. More concretely

$$T_{\mu\nu} = \partial_\mu\phi\, \partial_\nu\phi + g_{\mu\nu}\left(-\frac{1}{2}\partial_\rho\phi\, \partial^\rho\phi - V(\phi)\right) \;. \tag{13}$$

Solve the equations of motion of the system trying an Ansatz of the form

$$\begin{aligned} \phi &= \phi_0 \;, \\ ds^2 &= -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \;, \end{aligned} \tag{14}$$

with $\phi_0 = \text{cst}$ and $a(t)$ being a function of time. How is the value $\phi = \phi_0$ determined? What function $a(t)$ solves the Einstein equations?

Exercise 6

Consider a field theory including a spin 2 field (metric) and a massive spin 3/2 field (gravitino) in the presence of a cosmological constant. The action of the theory is given by

$$S \;=\; S_g \;+\; S_\Lambda \;+\; S_\Psi \;+\; S_{\text{mass}} \;, \tag{15}$$

with

$$\begin{aligned} S_g &= \frac{1}{2\kappa^2}\int d^4x\, e\, e_a{}^\mu\, e_b{}^\nu\, R_{\mu\nu}{}^{ab} \;, \\ S_\Lambda &= -\frac{1}{2\kappa^2}\int d^4x\, e\, \Lambda \;, \\ S_\Psi &= -\frac{1}{2\kappa^2}\int d^4x\, e\, \bar\Psi_\mu\, \gamma^{\mu\nu\rho} D_\nu \Psi_\rho \;, \\ S_{\text{mass}} &= -\frac{1}{2\kappa^2}\int d^4x\, e\, m\, \bar\Psi_\mu\, \gamma^{\mu\nu}\Psi_\nu \;. \end{aligned} \tag{16}$$

The first contribution S_g is the usual Einstein-Hilbert action for the metric. The second contribution S_Λ is that of a cosmological constant Λ. The third contribution S_Ψ is the Rarita-Schwinger action for the gravitino field. Finally, the fourth contribution S_{mass} with constant m is just a mass term for the gravitino field.

Show that the above action S is invariant, to lowest order in fermions, under the local supersymmetry transformations

$$\begin{aligned} \delta_\epsilon e_\mu{}^a &= \frac{1}{2}\,\bar\epsilon\,\gamma^a\,\Psi_\mu \;, \\ \delta_\epsilon \Psi_\mu &= D_\mu\epsilon - g\,\gamma_\mu\,\epsilon \;, \end{aligned} \tag{17}$$

with (spinorial) sypersymmetry parameter $\epsilon_\alpha(x)$ provided two relations of the form

$$\Lambda = c_1\, c_2\, g^2 \quad , \quad m = c_2\, g \;, \tag{18}$$

hold with c_1 and c_2 being constants. More concretely:

a) Determine the values of c_1 and c_2.

b) Discuss the relation between supersymmetry and the sign of the cosmological constant.

Note: $\gamma^{\mu\nu\rho}\,\gamma_\rho = 2\,\gamma^{\mu\nu}$.

Note: $\bar\chi\,\gamma_{\mu_1\mu_2\ldots\mu_n}\,\lambda = t_n\,\bar\lambda\,\gamma_{\mu_1\mu_2\ldots\mu_n}\,\chi$ with $t_0 = t_3 = -t_1 = -t_2 = 1$.

Exercise 7

Let us consider two real scalar fields ϕ^1 and ϕ^2 serving as coordinates on a two-dimensional field space with metric

$$ds^2 = K_{ij}(\phi)\, d\phi^i\, d\phi^j = \frac{1}{(\phi^2)^2}\Big((d\phi^1)^2 + (d\phi^2)^2\Big) \;. \tag{19}$$

Assuming a flat Minkowski space-time $g_{\mu\nu} = \eta_{\mu\nu}$ and taking the scalar fields to be only a function of time, *i.e.* $\phi^i = \phi^i(t)$:

a) Show that the action for the scalar fields

$$S = \int d^4x \left(-\frac{1}{2} K_{ij}\, \partial_\mu \phi^i\, \partial^\mu \phi^j \right) , \tag{20}$$

takes the simple form

$$S = \frac{1}{2} \int dt\, \frac{1}{(\phi^2)^2} \left((\dot{\phi}^1)^2 + (\dot{\phi}^2)^2 \right) , \tag{21}$$

where we have denoted $\dot{\phi}^i = \dfrac{d\phi^i}{dt}$.

b) Show that the Euler-Lagrange equations of motion are given by

$$\ddot{\phi}^1 - \frac{2}{\phi^2}\, \dot{\phi}^1\, \dot{\phi}^2 = 0 \qquad \text{and} \qquad \ddot{\phi}^2 + \frac{1}{\phi^2} \left((\dot{\phi}^1)^2 - (\dot{\phi}^2)^2 \right) = 0 \ . \tag{22}$$

c) Show that the Euler-Lagrange equations (22) can be expressed as a geodesic equation in field space

$$\ddot{\phi}^i + \Gamma_{jk}{}^i\, \dot{\phi}^j\, \dot{\phi}^k = 0 \ , \tag{23}$$

in terms of the Christoffel symbols in field space

$$\Gamma_{jk}{}^i = \tfrac{1}{2} K^{il} \left(\partial_j K_{lk} + \partial_k K_{jl} - \partial_l K_{jk} \right) \ . \tag{24}$$

d) Can you identify the coset space SL(2)/SO(2) in (19) upon a suitable field redefinition (change of coordinates in field space)?

Exercise 8

Let us consider the group of diffeomorphisms in a $(D+1)$-dimensional theory of gravity with space-time coordinates $x^M = (x^\mu, z)$. These diffeomorphisms are given by transformations on the metric of the form

$$\delta_{\widehat{\xi}}\, \widehat{g}_{MN} = \widehat{\xi}^P\, \partial_P \widehat{g}_{MN} + \widehat{g}_{MP}\, \partial_N \widehat{\xi}^P + \widehat{g}_{PN}\, \partial_M \widehat{\xi}^P \tag{25}$$

where

$$\widehat{\xi}^M(x,z) = \left(\widehat{\xi}^\mu \, , \, \widehat{\xi}^z \right) \ , \tag{26}$$

is the $(D+1)$-dimensional diffeomorphism parameter and

$$\widehat{\xi}^\mu = \xi^\mu(x) \qquad , \qquad \widehat{\xi}^z(x,z) = \theta(x) + c\, z \ , \tag{27}$$

with $\theta(x)$ being an arbitrary function of x^μ and c being a constant. In addition to the above diffeomorphisms, let us also consider scaling transformations on the metric of the form

$$\delta_a\, \widehat{g}_{MN} = 2\, a\, \widehat{g}_{MN} \ , \tag{28}$$

and a being a real constant.

a) Using the Kaluza-Klein Ansatz for the $(D+1)$-dimensional metric

$$\widehat{g}_{MN} = \begin{pmatrix} e^{2\alpha\phi}\, g_{\mu\nu} + e^{2\beta\phi}\, A_\mu\, A_\nu & e^{2\beta\phi}\, A_\mu \\ e^{2\beta\phi}\, A_\mu & e^{2\beta\phi} \end{pmatrix} , \tag{29}$$

with $\beta = -(D-2)\,\alpha$, show that (25) and (28) induce transformations $\delta = \delta_{\hat{\xi}} + \delta_a$ on the D-dimensional fields of the form

$$\begin{aligned}
\delta\phi \quad &= \quad \xi^\rho\,\partial_\rho\phi - \frac{1}{(D-2)\alpha}\,(c+a)\;,\\
\delta A_\mu \quad &= \quad \xi^\rho\,\partial_\rho A_\mu + A_\rho\,\partial_\mu\xi^\rho + \partial_\mu\theta - c\,A_\mu\;,\\
\delta g_{\mu\nu} \quad &= \quad \xi^\rho\,\partial_\rho g_{\mu\nu} + g_{\mu\rho}\,\partial_\nu\xi^\rho + g_{\rho\nu}\,\partial_\mu\xi^\rho + \frac{2}{(D-2)}\Big[c + a\,(D-1)\Big]\,g_{\mu\nu}\;.
\end{aligned} \tag{30}$$

b) Discuss the transformations arising upon the particular choices of parameters $a = -\dfrac{c}{(D-1)}$ and $a = -c$.

Exercise 9

Let us consider a Maxwell field $\widehat{B}_M$ in $D+1$ dimensions, with a $(D+1)$-dimensional index $M = \mu \oplus z$ splitting into a D-dimensional index μ and an additional direction z. The $(D+1)$-dimensional Maxwell field $\widehat{B}_M$ can then be decomposed as

$$\widehat{B}_M = (\,\widehat{B}_\mu\,,\,\widehat{B}_z\,) \equiv (\,B_\mu(x)\,,\,\chi(x)\,)\;. \tag{31}$$

Using the Kaluza-Klein Ansatz for the $(D+1)$-dimensional frame and its inverse

$$\widehat{e}_M{}^A = \begin{pmatrix} e^{\alpha\phi}\,e_\mu{}^a & e^{\beta\phi}\,A_\mu \\ 0_{1\times 4} & e^{\beta\phi} \end{pmatrix} \quad , \quad \widehat{e}_A{}^N = \begin{pmatrix} e^{-\alpha\phi}\,e_a{}^\nu & -e^{-\alpha\phi}\,A_a \\ 0_{1\times 4} & e^{-\beta\phi} \end{pmatrix}\;, \tag{32}$$

where $A_a \equiv e_a{}^\nu\,A_\nu$:

- a) Show that $\widehat{e}_M{}^A\,\widehat{e}_A{}^N = \delta_M{}^N$.
- b) Compute $\widehat{g}_{MN} = \widehat{e}_M{}^A\,\widehat{e}_N{}^B\,\widehat{\eta}_{AB}$, where $\widehat{\eta}_{AB}$ is the $(D+1)$-dimensional Minkowski metric, and show that the result agrees with (29).
- c) Show that, when $\beta = -(D-2)\,\alpha$, the $(D+1)$-dimensional Maxwell action

$$S_{\widehat{B}} = \int d^{D+1}x\,\sqrt{-|\widehat{g}|}\,\left(-\frac{1}{4}\,\widehat{F}_{MN}\,\widehat{F}^{MN}\right) = \int d^{D+1}x\;\widehat{e}\,\left(-\frac{1}{4}\,\widehat{F}_{AB}\,\widehat{F}^{AB}\right)\;, \tag{33}$$

reduces to a D-dimensional Maxwell-scalar action of the form

$$\begin{aligned}
S_{\widehat{B}} \quad &= \quad (2\pi L)\int d^Dx\; e\;\left(-\frac{1}{4}\,e^{-2\alpha\phi}\,\mathcal{F}_{ab}\,\mathcal{F}^{ab} - \frac{1}{2}\,e^{2(D-2)\alpha\phi}\,\partial_a\chi\,\partial^a\chi\right)\;,\\
&= \quad (2\pi L)\int d^Dx\;\sqrt{-|g|}\;\left(-\frac{1}{4}\,e^{-2\alpha\phi}\,\mathcal{F}_{\mu\nu}\,\mathcal{F}^{\mu\nu} - \frac{1}{2}\,e^{2(D-2)\alpha\phi}\,\partial_\mu\chi\,\partial^\mu\chi\right)\;,
\end{aligned} \tag{34}$$

with L being the radius of the (circle) z-direction, and where we have defined

$$\partial_a \equiv e_a{}^\nu\,\partial_\nu \quad , \quad \mathcal{F}_{ab} \equiv F_{ab} - 2\,\partial_{[a}\chi\,A_{b]}\;. \tag{35}$$